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Constrained dynamical systems, robust model reference adaptive control, and unreliable reference signals

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ABSTRACT

In this paper, we address the robust control design problem for nonlinear dynamical systems tracking unreliable reference signals. Specifically, we present robust model reference adaptive control laws that guarantee uniform ultimate boundedness of the trajectory tracking error for nonlinear plants that are affected by matched, unmatched, and parametric uncertainties, and are subject to constraints on the state space and the measured output. These control laws guarantee satisfactory results even in case the reference trajectory or the reference output signal do not verify the given constraints and hence, may draw the plant's trajectory or measured output outside their constraint sets. A numerical example involving the attitude control of a spacecraft illustrates the feasibility of the theoretical results presented.

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Barrier Lyapunov functions; constraints on the state space; model reference adaptive control; *e*-modification; projection operator

1. Introduction

Robust model reference adaptive control provides a formidable framework to design control laws for nonlinear plants, whose dynamics is partly unknown and affected by matched, unmatched, and parametric uncertainties. Specifically, robust model reference adaptive control laws are able to steer the trajectory of a nonlinear plant towards a given reference trajectory and guarantee uniformly ultimately bounded tracking errors despite external disturbances and uncertainties in the dynamical model (Ioannou & Fidan, 2006, Chapter 5; Lavretsky & Wise, 2012, Chapter 11). In the absence of unmatched uncertainties, model reference adaptive control laws also guarantee uniform asymptotic convergence of the closed-loop system's trajectory tracking error dynamics (Lavretsky & Wise, 2012, pp. 281–285).

Constraints on the state space and the measured output, which are imposed by the work environment, the required levels of performance, and the output sensors' operative ranges, add substantial complexity to the trajectory tracking problem. Control laws that guarantee satisfactory trajectory tracking or output tracking in the presence of constraints have been designed employing invariant set theory (Blanchini, 1991; Burger & Guay, 2010; Raković, Mayne, Kerrigan, & Kouramas, 2005), optimal control theory (Ko & Bitmead, 2007; Mayne & Schroeder, 1994; Park, Lee, Park, & Choi, 2012), receding horizon control and model predictive control (Boccia, Grne, & Worthmann, 2014; Mayne, Rawlings, Rao, & Scokaert, 2000; Scokaert & Rawlings, 1998; Sznaier & Suarez, 2001; Yoo, Lee, & Han, 2012), reference governors and explicit reference governors (Bemporad, 1998; Bemporad, Casavola, & Mosca, 1998; Garone, Nicotra, & Ntogramatzidis, 2018; Garone & Nicotra, 2016; Kolmanovsky, Garone, & Cairano, 2014), barrier Lyapunov functions (Liu, Lu, Li, & Tong, 2017), and restricted potential functions (Arabi, Gruenwald, Yucelen, & Nguyen, 2018). In particular, barrier Lyapunov functions and restricted potential functions have been employed to design adaptive control laws (Arabi et al., 2018; Bechlioulis & Rovithakis, 2009, 2010; Han, Ha, & Lee, 2016; Hosseinzadeh & Yazdanpanah, 2015; Hu, Shao, & Guo, 2018; Lozano-Leal, Collado, & Mondie, 1990; Nguyen, 2012; Pomet & Praly, 1992), backstepping control laws (Guo & Wu, 2014; Ngo, Mahony, & Jiang, 2005; Tee & Ge, 2011; Tee, Ge, & Tay, 2009), variable structure control laws (Dinuzzo, 2009; Ferrara, Incremona, & Rubagotti, 2014; Innocenti & Falorni, 1998; L'Afflitto & Mohammadi, 2017; Rubagotti, Raimondo, Ferrara, & Magni, 2011), and neural networks (He, Yin, & Sun, 2017; Zhao, Song, Ma, & He, 2018).

Optimal control frameworks, such as the receding horizon control and the model predictive control, employ Lagrange multipliers to recast constrained control synthesis problems as unconstrained optimal control problems. However, in applications involving nonlinear dynamical systems or non-convex constraint sets, solutions of the resulting unconstrained optimal control problems may not exist or could not be computed by existing numerical solvers (Goodwin et al., 2012). The reference governor approach guarantees satisfactory trajectory tracking by augmenting any control law designed to pursue the unconstrained trajectory tracking problem. In case the reference signals violate the system's constraints, explicit reference governors generate alternative reference signals that approximate the original ones, and verify the given constraints. Barrier Lyapunov functions and restricted potential functions are positive-definite on the constraint set, grow to infinity whenever their arguments approach the boundaries of the constraint set, and their total derivatives along the closed-loop system's vector field are negative on some subset of the constraint set. Thus, by applying

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sufficient conditions for uniform asymptotic stability, boundedness, or uniform ultimate boundedness, barrier Lyapunov functions and restricted potential functions certify that the closed-loop system's trajectory verifies the given constraints and satisfactorily tracks some reference signal. However, existing control synthesis methods that employ barrier Lyapunov functions and restricted potential functions to tackle the constrained trajectory tracking problem rely on the assumption that reference signals verify the constraints on the plant state and the measured output, and their effectiveness is not proved in case this assumption is not verified.

In this paper, we provide two sets of model reference adaptive control laws to steer constrained plants affected by matched, unmatched, and parametric uncertainties, so that the closed-loop system's trajectory and the measured output track given reference signals with uniformly ultimately bounded error. Despite existing frameworks that employ barrier Lyapunov functions or restricted potential functions, the proposed control laws guarantee satisfactory results even in case the reference signals violate the constraints on the plant trajectory and the measured output. The difficulty of following an unreliable reference signal with sufficiently small error lays in the fact that it may draw the closed-loop system's trajectory and measured output outside their constraint sets. We overcame this difficulty not only by employing restricted potential functions, but also by making conservative assumptions on the component of the reference signal's velocity that is orthogonal to the boundary of the constraint set. Despite existing approaches based on an optimal control framework, the control laws presented in this paper are suitable to address trajectory tracking problems involving nonlinear dynamical systems and non-convex constraints. Moreover, the proposed framework can be employed to effectively complement those approaches, such as the explicit reference governor, that entrust the guidance system with the tasks of modifying in real time the planned trajectory, should it be about to violate unforeseen constraints.

Assuming that the reference signals verify the constraints on the plant state and the measured output, we specialise our results to provide adaptive control laws, which guarantee that the closed-loop system's trajectory tracking error lays within predefined bounds at all time. These results are advantageous, since classical model reference adaptive control does not allow to impose *directly* any constraints on the trajectory tracking error. Indeed, classical model reference adaptive control laws only allow to bound *indirectly* the trajectory tracking error by tuning both the adaptive rates and the matrix gains obtained as solutions of a Lyapunov equation. In the absence of constraints, the proposed control laws reduce to known model reference adaptive controls for trajectory tracking (Kreisselmeier & Narendra, 1982; Narendra & Annaswamy, 1987).

This paper is organised as follows. In Section 2, we present the mathematical background and recall some fundamental results. In Section 3, we present a robust adaptive control law for constrained trajectory tracking, which is based on the e-modification of the classical model reference adaptive control (Narendra & Annaswamy, 1987). In Section 4, we present a robust adaptive control law for constrained output tracking, which employs a Lipschitz continuous projection operator (Kreisselmeier & Narendra, 1982; Pomet & Praly, 1992) to enforce prescribed bounds of the adaptive gains. In Section 5, we illustrate the feasibility of the proposed approaches by means of a numerical example involving the attitude control problem for a spacecraft, whose payload must not be pointed in some directions. Lastly, in Section 6, we draw conclusions and outline future research plans.

2. Notation, definitions, and mathematical preliminaries

In this section, we establish notation, definitions, and review some basic results. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ the set of $n \times m$ real matrices, and $\mathcal{B}_\varepsilon(x) \subset \mathbb{R}^n$ the *open ball centred at x with radius* ε. The *interior* of the set $\mathcal{C} \subset \mathbb{R}^n$ is denoted by $\mathring{\mathcal{C}}$, the *boundary* of $\mathcal{C}$ is denoted by $\partial\mathcal{C}$, and the *closure* of $\mathcal{C}$ is denoted by $\overline{\mathcal{C}}$. The *identity matrix* in $\mathbb{R}^{n\times n}$ is denoted by I_n or I, the *zero $n \times m$ matrix* in $\mathbb{R}^{n\times m}$ is denoted by $0_{n\times m}$ or 0, the *transpose* of $B \in \mathbb{R}^{n\times m}$ is denoted by B^{T}, and the *trace* of $A \in \mathbb{R}^{n\times n}$ is denoted by $\mathrm{tr}(A)$. We write $\|\cdot\|$ both for the *Euclidean vector norm* and the corresponding *equi-induced matrix norm*, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, and $\mathrm{spec}(A)$ for the *spectrum* of A. The *Fréchet derivative* of the continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$ at $x \in \mathcal{D} \subseteq \mathbb{R}^n$ is denoted by $h'(x) \triangleq \partial h(x)/\partial x$.

In the following, we recall both the definition of uniform ultimate boundedness and Lyapunov-like sufficient conditions for this form of stability. For the statement of these results, consider the nonlinear time-varying dynamical system

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0$, $f : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}^n$ is jointly continuous in its arguments, $f(t, \cdot)$ is locally Lipschitz continuous in x uniformly in t for all t in compact subsets of $t \in [t_0, \infty)$, and $0 = f(t, 0)$, $t \geq t_0$.

Definition 2.1 (Haddad & Chellaboina, 2008, Definition 4.4)**:** The nonlinear time-varying dynamical system (1) is *uniformly ultimately bounded* with ultimate bound $\varepsilon > 0$ if there exists $\gamma > 0$ independent of t_0 such that, for every $\delta \in (0, \gamma)$, there exists $T = T(\delta, \varepsilon) \geq 0$, independent of t_0, such that $x_0 \in \mathcal{B}_\delta(0)$ implies $x(t) \in \mathcal{B}_\varepsilon(0)$, $t \geq t_0 + T$.

Theorem 2.1 (Haddad & Chellaboina, 2008, Corollary 4.2)**:** *Consider the nonlinear time-varying dynamical system* (1) *and assume that there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that*

$$\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|), \quad (t, x) \in [t_0, \infty) \times \mathcal{D}, \tag{2}$$

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t, x) \leq -W(x), \quad (t, x) \in [t_0, \infty) \times \mathcal{Q}, \tag{3}$$

where $W : \mathcal{D} \to \mathbb{R}$ is continuous, $W(x) > 0$ for all $x \in \mathcal{Q}$, $\mathcal{Q} \triangleq \{x \in \mathcal{D} : \|x\| > \mu\}$, and $\mu > 0$ is such that $\mathcal{B}_{\alpha^{-1}(\eta)}(0) \subset \mathcal{D}$ for some $\eta > \beta(\mu)$. Then, (1) *is uniformly ultimately bounded with bound $\varepsilon = \alpha^{-1}(\eta)$.*

The problem of constraining the solutions of ordinary differential equations within open balls of given radius can be addressed by introducing a Lipschitz continuous form of the projection operator (Kreisselmeier & Narendra, 1982; Pomet & Praly, 1992) to partly modify the system's dynamics. To define this projection operator, let $\overline{\theta} > 0$ and $\gamma \in (0,1)$, and consider the continuously differentiable convex function

$$f_c(\theta) \triangleq \frac{(1+\gamma)\|\theta\|^2 - \overline{\theta}^2}{\gamma\overline{\theta}^2}, \quad \theta \in \mathbb{R}^n. \tag{4}$$

Definition 2.2 (Lavretsky & Wise, 2012, pp. 332, 337)**:** Consider the continuously differentiable function $f_c(\cdot)$ given by (4). The *vector projection operator* $\mathrm{proj} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\mathrm{proj}(\theta, y) \triangleq \begin{cases} y - \dfrac{\Gamma\left[f_c'(\theta)\right]^{\mathrm{T}} f_c'(\theta)}{\|f_c'(\theta)\|_\Gamma^2} y f_c(\theta), & \text{if } f_c(\theta) > 0 \text{ and } f_c'(\theta)y > 0, \\ y, & \text{otherwise}, \end{cases} \tag{5}$$

where $\Gamma \in \mathbb{R}^{n\times n}$ is symmetric and positive-definite and $\|x\|_\Gamma \triangleq [x^{\mathrm{T}}\Gamma x]^{1/2}$, $x \in \mathbb{R}^n$. The *matrix projection operator* is defined as

$$\mathrm{Proj}(\Theta, Y) \triangleq \left[\mathrm{proj}(\theta_1, y_1), \ldots, \mathrm{proj}(\theta_m, y_m)\right], \tag{6}$$

where $\Theta = [\theta_1, \ldots, \theta_m] \in \mathbb{R}^{n\times m}$ and $Y = [y_1, \ldots, y_m] \in \mathbb{R}^{n\times m}$.

Consider the ordinary differential equation

$$\dot{\theta}(t) = y(t), \quad \theta(t_0) = \theta_0, \quad t \geq t_0, \tag{7}$$

where $y : [t_0, \infty) \to \mathbb{R}^n$ is piece-wise continuous. The next result shows how the vector projection operator can be employed to constrain the solution of (7) so that $\theta(t) \in \overline{\mathcal{B}}_{\overline{\theta}}(0)$, $t \geq t_0$.

Theorem 2.2: *Consider the dynamical system*

$$\dot{\theta}(t) = \mathrm{proj}(\theta(t), y(t)), \quad \theta(t_0) = \theta_0, \quad t \geq t_0, \tag{8}$$

where $\mathrm{proj}(\cdot,\cdot)$ *is given by* (5). *If* $\|\theta_0\| \leq \overline{\theta}[1+\gamma]^{-(1/2)}$, *then* $\|\theta(t)\| \leq \overline{\theta}$, $t \geq t_0$.

Proof: The result is a direct consequence of Lemma 11.4 of Lavretsky and Wise (2012). ∎

3. Model reference adaptive control and state constraints

In this section, we address the trajectory tracking problem for nonlinear plants affected by matched and parametric uncertainties, and whose dynamics is partly unknown. Specifically, in the first part of this section, we present a model reference adaptive control law, which guarantees that the closed-loop system's trajectory always lays in a compact, simply connected constraint set and tracks with uniformly ultimately bounded error the trajectory of a reference model. The proposed control law is effective even in case the reference model's trajectory is *not* entirely contained in the constraint set. Assuming that the reference model's trajectory is contained in the constraint set at all time, in the second part of this section we provide an adaptive control law that enforces prescribed bounds on the trajectory tracking error at all time and guarantees its uniform asymptotic convergence to zero.

Consider the nonlinear time-invariant plant

$$\dot{x}(t) = Ax(t) + B\Lambda\left[u(t) + \Theta^{\mathrm{T}}\Phi(x(t))\right], \quad x(t_0) = x_0, \quad t \geq t_0, \tag{9}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0$, denotes the *plant's trajectory*, $0 \in \mathcal{D}$, $u(t) \in \mathbb{R}^m$ denotes the *control input*, $A \in \mathbb{R}^{n\times n}$ is *unknown*, $B \in \mathbb{R}^{n\times m}$, $\Lambda \in \mathbb{R}^{m\times m}$ is diagonal, positive-definite, and *unknown*, $\Theta \in \mathbb{R}^{N\times m}$ is *unknown*, and the *regressor vector* $\Phi : \mathbb{R}^n \to \mathbb{R}^N$ is Lipschitz continuous. Both Λ and $\Theta^{\mathrm{T}}\Phi(x)$, $x \in \mathcal{D}$, capture the plant's matched and parametric uncertainties, such as malfunctions in the control system, and Λ is such that the pair $(A, B\Lambda)$ is controllable and $\Lambda_{\min} I_m \leq \Lambda$, for some $\Lambda_{\min} > 0$. Although the matrix A is unknown, in practical applications the structure of A is usually known and the controllability of the pair $(A, B\Lambda)$ can be verified (Lavretsky & Wise, 2012, pp. 281–282). Consider also the linear reference model

$$\dot{x}_{\mathrm{ref}}(t) = A_{\mathrm{ref}} x_{\mathrm{ref}}(t) + B_{\mathrm{ref}} r(t), \quad x_{\mathrm{ref}}(t_0) = x_{\mathrm{ref},0}, \quad t \geq t_0, \tag{10}$$

where $x_{\mathrm{ref}}(t) \in \mathcal{D}$, $t \geq t_0$, denotes the *reference trajectory*, $A_{\mathrm{ref}} \in \mathbb{R}^{n\times n}$ is Hurwitz, $B_{\mathrm{ref}} \in \mathbb{R}^{n\times m}$, and the *reference input* $r(t) \in \mathbb{R}^m$ is piece-wise continuous and bounded, that is, $\|r(t)\| \leq r_{\max}$, $t \geq t_0$, where $r_{\max} \geq 0$. Lastly, let

$$e(t) \triangleq x(t) - x_{\mathrm{ref}}(t), \quad t \geq t_0, \tag{11}$$

denote the *trajectory tracking error* and consider the compact, simply connected *constraint set*

$$\mathcal{C} \triangleq \{x \in \mathcal{D} : h(x) \geq 0\}, \tag{12}$$

where $h : \mathcal{D} \to \mathbb{R}$ is continuously differentiable and $h(0) > 0$; the interior of $\mathcal{C}$, that is, $\mathring{\mathcal{C}} = \{x \in \mathcal{D} : h(x) > 0\}$, is non-empty.

The next theorem is the main result of this section and provides a robust model reference adaptive control law for $u(\cdot)$, so that the plant trajectory $x(\cdot)$ lays in the interior of the constraint set $\mathcal{C}$ at all time, that is, $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, and tracks with uniformly ultimately bounded error the reference trajectory $x_{\mathrm{ref}}(\cdot)$, even in case $x_{\mathrm{ref}}(t) \notin \mathcal{C}$ for some $t \in [t_0, \infty)$. For the statement of this result, consider the algebraic Lyapunov equation

$$0 = A_{\mathrm{ref}}^{\mathrm{T}} P + P A_{\mathrm{ref}} + Q, \tag{13}$$

where $A_{\mathrm{ref}} \in \mathbb{R}^{n\times n}$ is Hurwitz, $Q \in \mathbb{R}^{n\times n}$ is symmetric and positive-definite, and $P \in \mathbb{R}^{n\times n}$; since A_{ref} is Hurwitz and Q is symmetric and positive-definite, there exists a unique symmetric and positive-definite matrix P that verifies (13) (Bernstein, 2009, Proposition 11.9.5). Furthermore, let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of their arguments, respectively, let $\rho_{\mathcal{C}} \triangleq \max_{x\in\mathcal{C}} \|x\|$, and let $\overline{h}_{\mathrm{d}} \triangleq \max_{x\in\mathcal{C}} \|h'(x)\|$; it follows from Theorem 2.13 of Haddad and Chellaboina (2008) that both $\rho_{\mathcal{C}}$ and $\overline{h}_{\mathrm{d}}$ exist and are finite, since $\mathcal{C}$ is compact and both $\|\cdot\|$ and $h'(\cdot)$ are continuous on $\mathcal{C}$.

Lastly, let $s(t,x) \in \mathcal{D}$, $(t,x) \in [t_0,\infty) \times \mathcal{D}$, denote the *solution of* (9) *with initial condition* x*, evaluated at time* $t \geq t_0$ (Haddad & Chellaboina, 2008, p. 71), and

$$\kappa \triangleq \min_{x \in \mathcal{S}_{x_0}} h(x), \tag{14}$$

where $\mathcal{S}_{x_0} \triangleq \{x \in \mathring{\mathcal{C}} : x = s(t,x_0), \text{for some } t \geq t_0\}$ denotes the plant's *path*; since $\mathcal{S}_{x_0}$ is a compact subset of $\mathcal{D}$ and $h(\cdot)$ is continuous on $\mathcal{S}_{x_0}$, $\kappa > 0$ is well defined (Haddad & Chellaboina, 2008, Theorem 2.13).

Theorem 3.1: *Consider the nonlinear plant* (9) *with* $x_0 \in \mathring{\mathcal{C}}$*, the constraint set* (12)*, the reference model* (10)*, and the adaptive laws*

$$\dot{\hat{K}}_x(t) = -\Gamma_x x(t) \frac{e^{\mathrm{T}}(t)P}{h(x(t))}\left[I_n - e(t)\frac{h'(x(t))}{2h(x(t))}\right]B - 2\sigma_x(e(t))\hat{K}_x(t),$$
$$\hat{K}_x(t_0) = \hat{K}_{x,0}, \quad t \geq t_0, \tag{15}$$

$$\dot{\hat{K}}_r(t) = -\Gamma_r r(t) \frac{e^{\mathrm{T}}(t)P}{h(x(t))}\left[I_n - e(t)\frac{h'(x(t))}{2\,h(x(t))}\right]B - 2\sigma_r(e(t))\hat{K}_r(t), \quad \hat{K}_r(t_0) = \hat{K}_{r,0}, \tag{16}$$

$$\dot{\hat{\Theta}}(t) = \Gamma_\Theta \Phi(x(t)) \frac{e^{\mathrm{T}}(t)P}{h(x(t))}\left[I_n - e(t)\frac{h'(x(t))}{2\,h(x(t))}\right]B - 2\sigma_\Theta(e(t))\hat{K}_\Theta(t), \quad \hat{\Theta}(t_0) = \hat{\Theta}_0, \tag{17}$$

where $\sigma_x(e) \triangleq \overline{\sigma}_x \|B^{\mathrm{T}}Pe\|$*,* $e(\cdot)$ *is given by* (11)*,* $\sigma_r(e) \triangleq \overline{\sigma}_r \|B^{\mathrm{T}}Pe\|$*,* $\sigma_\Theta(e) \triangleq \overline{\sigma}_\Theta \|B^{\mathrm{T}}Pe\|$*,* $\overline{\sigma}_x, \overline{\sigma}_r, \overline{\sigma}_\Theta > 0$*, the matrices* $\Gamma_x \in \mathbb{R}^{n\times n}$*,* $\Gamma_r \in \mathbb{R}^{m\times m}$*, and* $\Gamma_\Theta \in \mathbb{R}^{N\times N}$ *are symmetric and positive-definite, and* P *is the symmetric positive-definite solution of the Lyapunov equation* (13)*. If* $x_{\mathrm{ref},0} \in \mathring{\mathcal{C}}$*,*

$$\frac{\lambda_{\max}(P)\overline{h}_{\mathrm{d}}\,[\|A_{\mathrm{ref}}\|\rho_{\mathcal{C}} + \|B_{\mathrm{ref}}\|r_{\max}]}{\lambda_{\min}(Q)} < \kappa, \tag{18}$$

where κ *is given by* (14)*, and there exist* $K_x \in \mathbb{R}^{n\times m}$ *and* $K_r \in \mathbb{R}^{m\times m}$ *such that*

$$A_{\mathrm{ref}} = A + B\Lambda K_x^{\mathrm{T}}, \tag{19}$$

$$B_{\mathrm{ref}} = B\Lambda K_r^{\mathrm{T}}, \tag{20}$$

then the trajectory of (9) *with* $u = \phi(x, \hat{K}_x, \hat{K}_r, \hat{\Theta})$*, where*

$$\phi(t,x,\hat{K}_x,\hat{K}_r,\hat{\Theta}) = \hat{K}_x^{\mathrm{T}}x + \hat{K}_r^{\mathrm{T}}r(t) - \hat{\Theta}^{\mathrm{T}}\Phi(x),$$
$$(t,x,\hat{K}_x,\hat{K}_r,\hat{\Theta}) \in [t_0,\infty) \times \mathring{\mathcal{C}} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times n}, \tag{21}$$

is such that $x(t) \in \mathring{\mathcal{C}}$*,* $t \geq t_0$*. Moreover, there exist* $\varepsilon > 0$ *and* $\gamma > 0$ *such that, for every* $\delta \in (0,\gamma)$*, there exists a finite-time* $T = T(\delta,\varepsilon) \geq 0$ *such that* $e(t_0) \in \mathcal{B}_\delta(0)$ *implies*

$$e(t) \in \mathcal{B}_\varepsilon(0), \quad t \geq t_0 + T. \tag{22}$$

Proof: Let $x(\cdot)$ denote the solution of (9) with control law (21). Firstly, we apply Theorem 2.1 to prove that if $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, then the trajectory tracking error is uniformly ultimately bounded. Successively, we make a contradiction argument to prove that if $x(t_0) \in \mathring{\mathcal{C}}$, then $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$.

Let $\Delta K_x \triangleq \hat{K}_x - K_x$, $\Delta K_r \triangleq \hat{K}_r - K_r$, and $\Delta\Theta \triangleq \hat{\Theta} - \Theta$. It follows from (11), (9) with control law (21), and (10) that

$$\dot{e}(t) = A_{\mathrm{ref}}e(t) + B\Lambda\left[\Delta K_x^{\mathrm{T}}(t)x(t) + \Delta K_r^{\mathrm{T}}(t)r(t) - \Delta\Theta^{\mathrm{T}}(t)\Phi(x(t))\right],$$
$$e(t_0) = x_0 - x_{\mathrm{ref},0}, \quad t \geq t_0, \tag{23}$$

and it follows from (19) and (20) that (9) with control law (21) is equivalent to

$$\dot{x}(t) = [A_{\mathrm{ref}}x(t) + B_{\mathrm{ref}}r(t)] + B\Lambda\left[\Delta K_x^{\mathrm{T}}(t)x(t) + \Delta K_r^{\mathrm{T}}(t)r(t) - \Delta\Theta^{\mathrm{T}}(t)\Phi(x(t))\right],$$
$$x(t_0) = x_0. \tag{24}$$

Consider the positive-definite, decrescent function

$$V(t,e,\Delta K_x,\Delta K_r,\Delta\Theta) \triangleq \frac{e^{\mathrm{T}}Pe}{h(x(t))} + \mathrm{tr}\left(\left[\Delta K_x^{\mathrm{T}}\Gamma_x^{-1}\Delta K_x + \Delta K_r^{\mathrm{T}}\Gamma_r^{-1}\Delta K_r + \Delta\Theta^{\mathrm{T}}\Gamma_\Theta^{-1}\Delta\Theta\right]\Lambda\right)$$
$$\times\ (t,e,\Delta K_x,\Delta K_r,\Delta\Theta) \in [t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times m}, \tag{25}$$

where $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, denotes the solution of (24). It follows from (23) and (24) that

$$\dot{V}(t,e,\Delta K_x,\Delta K_r,\Delta\Theta) = -\frac{e^{\mathrm{T}}Qe}{h(x(t))} - \frac{e^{\mathrm{T}}Pe}{h^2(x(t))}h'(x(t))\,[A_{\mathrm{ref}}x(t) + B_{\mathrm{ref}}r(t)]$$
$$+ \mathrm{tr}\left(\Delta K_x^{\mathrm{T}}\left[2\Gamma_x^{-1}\dot{\hat{K}}_x(t) + 2x(t)\frac{e^{\mathrm{T}}P}{h(x(t))} \times \left(I_n - e\frac{h'(x(t))}{2\,h(x(t))}\right)B\right]\Lambda\right)$$
$$+ \mathrm{tr}\left(\Delta K_r^{\mathrm{T}}\left[2\Gamma_r^{-1}\dot{\hat{K}}_r(t) + 2r(t)\frac{e^{\mathrm{T}}P}{h(x(t))} \times \left(I_n - e\frac{h'(x(t))}{2\,h(x(t))}\right)B\right]\Lambda\right)$$
$$+ \mathrm{tr}\left(\Delta\Theta^{\mathrm{T}}\left[2\Gamma_\Theta^{-1}\dot{\hat{\Theta}}(t) - 2\Phi(x(t))\frac{e^{\mathrm{T}}P}{h(x(t))} \times \left(I_n + e\frac{h'(x(t))}{2\,h(x(t))}\right)B\right]\Lambda\right),$$
$$(t,e,\Delta K_x,\Delta K_r,\Delta\Theta) \in [t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times m}, \tag{26}$$

and it follows from (15) and (17) that

$$\begin{aligned}
&\dot{V}(t, e, \Delta K_x, \Delta K_r, \Delta\Theta)\\
&\leq -\lambda_{\min}(Q)h^{-1}(x(t))\|e\|^2\\
&\quad + \lambda_{\max}(P)h^{-2}(x(t))\|e\|^2\overline{h}_{\mathrm{d}}\left(\|A_{\mathrm{ref}}\|\rho_{\mathcal{C}} + \|B_{\mathrm{ref}}\|r_{\max}\right)\\
&\quad - 2\sigma_x(e)\mathrm{tr}\left(\Delta K_x^{\mathrm{T}}\Delta K_x\Lambda\right) - 2\sigma_x(e)\mathrm{tr}\left(\Delta K_x^{\mathrm{T}}K_x\Lambda\right)\\
&\quad - 2\sigma_r(e)\mathrm{tr}\left(\Delta K_r^{\mathrm{T}}\Delta K_r\Lambda\right) - 2\sigma_r(e)\mathrm{tr}\left(\Delta K_r^{\mathrm{T}}K_r\Lambda\right)\\
&\quad - 2\sigma_\Theta(e)\mathrm{tr}\left(\Delta\Theta^{\mathrm{T}}\Delta\Theta\Lambda\right) - 2\sigma_\Theta(e)\mathrm{tr}\left(\Delta\Theta^{\mathrm{T}}\Theta\Lambda\right), \quad (27)
\end{aligned}$$

along the trajectories of (23) and (15)–(17). Moreover, recalling that $|\mathrm{tr}(XY)| \leq \|X\|_{\mathrm{F}}\|Y\|_{\mathrm{F}}$, where $X \in \mathbb{R}^{n_1\times n_2}$ and $Y \in \mathbb{R}^{n_2\times n_1}$, it holds that

$$\begin{aligned}
&\dot{V}(t, e, \Delta K_x, \Delta K_r, \Delta\Theta)\\
&\leq -\lambda_{\min}(Q)h^{-1}(x(t))\|e\|^2\\
&\quad + \lambda_{\max}(P)h^{-2}(x(t))\|e\|^2\overline{h}_{\mathrm{d}}\left(\|A_{\mathrm{ref}}\|\rho_{\mathcal{C}} + \|B_{\mathrm{ref}}\|r_{\max}\right)\\
&\quad - 2\sigma_x(e)\|\Delta K_x\|_{\mathrm{F}}^2\Lambda_{\min} + 2\sigma_x(e)\|\Delta K_x\|_{\mathrm{F}}\|K_x\|_{\mathrm{F}}\|\Lambda\|_{\mathrm{F}}\\
&\quad - 2\sigma_r(e)\|\Delta K_r\|_{\mathrm{F}}^2\Lambda_{\min} + 2\sigma_r(e)\|\Delta K_r\|_{\mathrm{F}}\|K_r\|_{\mathrm{F}}\|\Lambda\|_{\mathrm{F}}\\
&\quad - 2\sigma_\Theta(e)\|\Delta\Theta\|_{\mathrm{F}}^2\Lambda_{\min} + 2\sigma_\Theta(e)\|\Delta\Theta\|_{\mathrm{F}}\|\Theta\|_{\mathrm{F}}\|\Lambda\|_{\mathrm{F}}\\
&\leq h^{-2}(x(t))\left[\lambda_{\max}(P)\overline{h}_{\mathrm{d}}\left(\|A_{\mathrm{ref}}\|\rho_{\mathcal{C}} + \|B_{\mathrm{ref}}\|r_{\max}\right)\right.\\
&\quad \left.- \lambda_{\min}(Q)h(x(t))\right]\|e\|^2\\
&\quad - 2\sigma_x(e)\Lambda_{\min}\left(\|\Delta K_x\|_{\mathrm{F}} - \frac{1}{2}\frac{\|K_x\|_{\mathrm{F}}\|\Lambda\|_{\mathrm{F}}}{\Lambda_{\min}}\right)^2\\
&\quad - 2\sigma_r(e)\Lambda_{\min}\left(\|\Delta K_r\|_{\mathrm{F}} - \frac{1}{2}\frac{\|K_r\|_{\mathrm{F}}\|\Lambda\|_{\mathrm{F}}}{\Lambda_{\min}}\right)^2\\
&\quad - 2\sigma_\Theta(e)\Lambda_{\min}\left(\|\Delta\Theta\|_{\mathrm{F}} - \frac{1}{2}\frac{\|\Theta\|_{\mathrm{F}}\|\Lambda\|_{\mathrm{F}}}{\Lambda_{\min}}\right)^2\\
&\quad + \left[\sigma_x(e)\|K_x\|_{\mathrm{F}}^2 + \sigma_r(e)\|K_r\|_{\mathrm{F}}^2 + \sigma_\Theta(e)\|\Theta\|_{\mathrm{F}}^2\right]\frac{\|\Lambda\|_{\mathrm{F}}^2}{2\Lambda_{\min}}. \quad (28)
\end{aligned}$$

Since $h(x) > 0$ for all $x \in \mathring{\mathcal{C}}$, it follows from (14) and (18) that

$$\dot{V}(t, e, \Delta K_x, \Delta K_r, \Delta\Theta) \leq -W(e, \Delta K_x, \Delta K_r, \Delta\Theta), \quad (t, e, \Delta K_x, \Delta K_r, \Delta\Theta) \in [t_0, \infty) \times \mathcal{Q}, \quad (29)$$

where

$$\begin{aligned}
&W(e, \Delta K_x, \Delta K_r, \Delta\Theta)\\
&\triangleq -\kappa^{-2}\left[\lambda_{\max}(P)\overline{h}_{\mathrm{d}}(\|A_{\mathrm{ref}}\|\rho_{\mathcal{C}}\right.\\
&\quad \left.+ \|B_{\mathrm{ref}}\|r_{\max}) - \lambda_{\min}(Q)\kappa\right]\|e\|^2\\
&\quad + 2\overline{\sigma}_x\|B^{\mathrm{T}}Pe\|\Lambda_{\min}\left(\|\Delta K_x\|_{\mathrm{F}} - \frac{1}{2}\frac{\|K_x\|_{\mathrm{F}}\|\Lambda\|_{\mathrm{F}}}{\Lambda_{\min}}\right)^2\\
&\quad + 2\overline{\sigma}_r\|B^{\mathrm{T}}Pe\|\Lambda_{\min}\left(\|\Delta K_r\|_{\mathrm{F}} - \frac{1}{2}\frac{\|K_r\|_{\mathrm{F}}\|\Lambda\|_{\mathrm{F}}}{\Lambda_{\min}}\right)^2\\
&\quad + 2\overline{\sigma}_\Theta\|B^{\mathrm{T}}Pe\|\Lambda_{\min}\left(\|\Delta\Theta\|_{\mathrm{F}} - \frac{1}{2}\frac{\|\Theta\|_{\mathrm{F}}\|\Lambda\|_{\mathrm{F}}}{\Lambda_{\min}}\right)^2\\
&\quad - \left[\overline{\sigma}_x\|K_x\|_{\mathrm{F}}^2 + \overline{\sigma}_r\|K_r\|_{\mathrm{F}}^2 + \overline{\sigma}_\Theta\|\Theta\|_{\mathrm{F}}^2\right]\frac{\|\Lambda\|_{\mathrm{F}}^2\|B^{\mathrm{T}}Pe\|}{2\Lambda_{\min}} \quad (30)
\end{aligned}$$

is positive for sufficiently large values of $\|e\|$, $\|\Delta K_x\|_{\mathrm{F}}$, $\|\Delta K_r\|_{\mathrm{F}}$, and $\|\Delta\Theta\|_{\mathrm{F}}$; an explicit expression for $\mathcal{Q} \subseteq \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times m}$ is omitted for brevity. Since (25) is positive-definite and decrescent, there exist class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that

$$\begin{aligned}
&\alpha\left(\left\|[\|e\|, \|\Delta K_x\|, \|\Delta K_r\|, \|\Delta\Theta\|]^{\mathrm{T}}\right\|\right)\\
&\quad \leq V(t, e, \Delta K_x, \Delta K_r, \Delta\Theta)\\
&\quad \leq \beta\left(\left\|[\|e\|, \|\Delta K_x\|, \|\Delta K_r\|, \|\Delta\Theta\|]^{\mathrm{T}}\right\|\right), \quad (31)
\end{aligned}$$

for all $(t, e, \Delta K_x, \Delta K_r, \Delta\Theta) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times m}$. Thus, it follows from (31), (29), and (30) that the conditions of Theorem 2.1 are verified and if $x(t) \in \mathring{\mathcal{C}}$, then the nonlinear time-invariant dynamical system given by (23) and (15)–(17) is uniformly ultimately bounded, that is, there exists $\varepsilon > 0$ and $\gamma > 0$ such that, for every $\delta \in (0, \gamma)$, there exists a finite-time $T = T(\delta, \varepsilon) \geq 0$ such that if $e(t_0) \in \mathcal{B}_\delta(0)$, then (22) is verified.

Next, we prove that if $x_0 \in \mathring{\mathcal{C}}$, then $x(t) \in \mathring{\mathcal{C}}$ for all $t \geq t_0$. Suppose *ad absurdum* that there exists a finite-time $\overline{T} > t_0$ such that $h(x(\overline{T})) = 0$ along the trajectory of (24), that is, such that $x(\overline{T}) \in \partial\mathcal{C}$. If $e(\overline{T}) = 0$, then it follows from (23) and (15)–(17) that $e(t) = e(\overline{T}) = 0$, $t \geq \overline{T}$, $\Delta K_x(t) = \Delta K_x(\overline{T})$, $\Delta K_r(t) = \Delta K_r(\overline{T})$, and $\Delta\Theta(t) = \Delta\Theta(\overline{T})$, and hence $V(t, e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t)) = V(\overline{T}, 0, \Delta K_x(\overline{T}), \Delta K_r(\overline{T}), \Delta\Theta(\overline{T}))$, $t \geq \overline{T}$, which is finite. However, it follows from (26) and l'Hôpital's theorem that either $\lim_{t\to\overline{T}} V(t, e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t)) = \infty$, or $\lim_{t\to\overline{T}} V(t, e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t)) = V(\overline{T}, e(\overline{T}), \Delta K_x(\overline{T}), \Delta K_r(\overline{T}), \Delta\Theta(\overline{T}))$ and $\lim_{t\to\overline{T}} |\dot{V}(t, e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t))| = \infty$, and either case contradicts the deduction that $V(t, e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t))$, $t \geq \overline{T}$, is constant and finite. Alternatively, if $e(\overline{T}) \neq 0$, then it follows from (25) that $\lim_{t\to\overline{T}} V(t, e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t)) = \infty$, However, it follows from (29) that $V(t, e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t))$, $t \geq t_0$, is uniformly bounded, which is a contradiction. Thus, if $x_0 \in \mathring{\mathcal{C}}$, then $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$. ■

Theorem 3.1 provides sufficient conditions for the plant trajectory $x(\cdot)$ to lay in the interior of the compact, simply connected constraint set $\mathcal{C}$ at all time and track the reference model's trajectory $x_{\mathrm{ref}}(\cdot)$ with uniformly ultimately bounded error, even in case $x_{\mathrm{ref}}(\cdot)$ does not always lay in $\mathcal{C}$. The positive-definite, decrescent function (25) is a *restricted potential function* (Arabi et al., 2018), since $V(\cdot, \cdot, \cdot, \cdot, \cdot)$ is employed to prove uniform ultimate boundedness of the nonlinear dynamical system given by (23) and (15)–(17), and if $\lim_{t\to\overline{T}} \mathrm{dist}(x(t), \partial\mathcal{C}) = 0$ uniformly in t_0 for some $\overline{T} >$

t_0, then $\lim_{t\to\overline{T}} V(t, e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t)) = \infty$ uniformly in t_0, where $\mathrm{dist}(\cdot,\cdot)$ denotes the distance of a vector from a set (Kreyszig, 1989, p. 16).

If the reference trajectory $x_{\mathrm{ref}}(\cdot)$ does not lay in the interior of the constraint set $\mathcal{C}$, then the feedback control law (21) must meet two competing objectives, namely guaranteeing that the trajectory tracking error $e(\cdot)$ is uniformly ultimately bounded and the plant trajectory $x(\cdot)$ lays in $\mathring{\mathcal{C}}$ at all time. To meet these objectives, Theorem 3.1 relies on the conservative assumption that (18) is satisfied. The parameter $\kappa > 0$, which is defined in (14) and appears on the right-hand side of (18), can only be estimated a priori and it follows from (29) and (30) that smaller values of κ imply smaller ultimate bounds on the trajectory tracking error. The left-hand side of (18) captures a conservative estimate of the reference trajectory's velocity normal to the boundary of the constraint set. Indeed, it follows from (10) that $h'(x(t))\dot{x}_{\mathrm{ref}}(t) = h'(x(t))[A_{\mathrm{ref}}x_{\mathrm{ref}}(t) + B_{\mathrm{ref}}r(t)]$, $t \geq t_0$, and if $x_{\mathrm{ref}}(t^*) \in \partial\mathcal{C}$, for some $t^* > t_0$, then $h'(x(t^*))\dot{x}_{\mathrm{ref}}(t^*) \leq \overline{h}_{\mathrm{d}}[\|A_{\mathrm{ref}}\|\rho_{\mathcal{C}} + \|B_{\mathrm{ref}}\|r_{\max}]$. Since $\|\dot{x}_{\mathrm{ref}}(t)\| \leq \|A_{\mathrm{ref}}\|\rho_{\mathcal{C}} + \|B_{\mathrm{ref}}\|r_{\max}$, $t \geq t_0$, it follows from (18) that smaller values of $[\|A_{\mathrm{ref}}\|\rho_{\mathcal{C}} + \|B_{\mathrm{ref}}\|r_{\max}]$, that is, slower reference signals, are necessary to allow smaller values of κ and hence, to guarantee smaller ultimate bounds on the trajectory tracking error, whenever the reference signal does not lay entirely in the constraint set. The left-hand side of (18) comprises the term $\lambda_{\max}(P)/\lambda_{\min}(Q)$. Now, note that if $z_{\mathrm{ref}} : [t_0, \infty) \to \mathbb{R}^n$ denotes the solution of

$$\dot{z}_{\mathrm{ref}}(t) = A_{\mathrm{ref}}z_{\mathrm{ref}}(t), \quad z_{\mathrm{ref}}(t_0) = x_{\mathrm{ref},0}, \quad t \geq t_0,$$

then it follows from (13) that $\|z_{\mathrm{ref}}(t)\| \leq [e^{-((\lambda_{\min}(Q)(t-t_0))/(\lambda_{\max}(P)))}\lambda_{\min}^{-1}(P)x_{\mathrm{ref},0}^{\mathrm{T}}Px_{\mathrm{ref},0}]^{1/2}$, $t \geq t_0$ (Haddad & Chellaboina, 2008, p. 178). Thus, smaller values of $(\lambda_{\max}(P))/(\lambda_{\min}(Q))$ are necessary to allow smaller values of κ and hence, to guarantee smaller ultimate bounds on the trajectory tracking error, whenever the reference signal does not lay entirely in the constraint set. However, smaller values of $\lambda_{\max}(P)/\lambda_{\min}(Q)$ also imply more rapid excursions of the reference model dynamics (10) during the transient period.

The effect of the proximity of the plant trajectory $x(\cdot)$ to the boundary of the constraint set $\partial\mathcal{C}$ on the adaptive control law (21) is captured by the term $h^{-1}(x(\cdot))$ in (15)–(17). Indeed, it follows from (15)–(17), the continuity of $h^{-1}(\cdot)$ on $\mathring{\mathcal{C}}$, and the continuity of the distance function $\mathrm{dist}(\cdot,\cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$ that for every $M > 0$, there exists $\delta(M) > 0$ such that if $\mathrm{dist}(x, \partial\mathcal{C}) < \delta$, then $\|\dot{\hat{K}}_x\|$, $\|\dot{\hat{K}}_r\|$, and $\|\dot{\hat{\Theta}}\| > M$. Therefore, smaller values of $\mathrm{dist}(x(\cdot), \partial\mathcal{C})$ imply larger values of $h^{-1}(x(\cdot))$ and hence of $\dot{\hat{K}}_x(\cdot)$, $\dot{\hat{K}}_r(\cdot)$, and $\dot{\hat{\Theta}}(\cdot)$. In practice, if the plant trajectory $x(\cdot)$ approaches the boundary of the constraint set $\partial\mathcal{C}$, then the adaptive gains experience rapid variations.

The next theorem specialises Theorem 3.1 to the case wherein $x_{\mathrm{ref}}(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$.

Theorem 3.2: *Consider the constraint set* (12), *the nonlinear plant* (9) *with* $x_0 \in \mathring{\mathcal{C}}$, *the reference model* (10), *the trajectory tracking error* (11), *the adaptive laws*

$$\dot{\hat{K}}_x(t) = -\Gamma_x x(t)\frac{e^{\mathrm{T}}(t)P}{h(x(t))}\left[I_n - e(t)\frac{h'(x(t))}{2h(x(t))}\right]B,$$
$$\hat{K}_x(t_0) = \hat{K}_{x,0}, \quad t \geq t_0, \tag{32}$$

$$\dot{\hat{K}}_r(t) = -\Gamma_r r(t)\frac{e^{\mathrm{T}}(t)P}{h(x(t))}\left[I_n - e(t)\frac{h'(x(t))}{2\,h(x(t))}\right]B,$$
$$\hat{K}_r(t_0) = \hat{K}_{r,0}, \tag{33}$$

$$\dot{\hat{\Theta}}(t) = \Gamma_\Theta\Phi(x(t))\frac{e^{\mathrm{T}}(t)P}{h(x(t))}\left[I_n - e(t)\frac{h'(x(t))}{2h(x(t))}\right]B,$$
$$\hat{\Theta}(t_0) = \hat{\Theta}_0, \tag{34}$$

and the feedback control law $\phi(\cdot,\cdot,\cdot,\cdot,\cdot)$ *given by* (21). *If* $x_{\mathrm{ref}}(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, *and there exist* $K_x \in \mathbb{R}^{n\times m}$ *and* $K_r \in \mathbb{R}^{m\times m}$ *such that* (19) *and* (20) *are verified, then the trajectory of* (9) *with* $u = \phi(t, x, \hat{K}_x, \hat{K}_r, \hat{\Theta})$, $(t, x, \hat{K}_x, \hat{K}_r, \hat{\Theta}) \in [t_0, \infty) \times \mathring{\mathcal{C}} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times n}$, *is such that* $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, *and* $\|x(t) - x_{\mathrm{ref}}(t)\| \to 0$ *as* $t \to \infty$ *uniformly in* t_0.

Proof: The proof of this result is similar to the proof of Theorem 3.1 and hence, is only briefly outlined. Consider the positive-definite, decrescent Lyapunov function (25). If $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, where $x(\cdot)$ denotes the solution of (9) with control law (21), then one can prove that

$$\dot{V}(t, e, \Delta K_x, \Delta K_r, \Delta\Theta) = -\frac{e^{\mathrm{T}}Qe}{h(x(t))},$$
$$(t, e, \Delta K_x, \Delta K_r, \Delta\Theta) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times m},$$

along the trajectory of the nonlinear dynamical system given by (23) and (32)–(34). Thus, uniform asymptotic convergence of the trajectory tracking error $e(\cdot)$ can be proved invoking Barbalat's lemma (Haddad & Chellaboina, 2008, Lemma 4.1). Successively, using a contradiction argument similar to the one presented in the proof of Theorem 3.1, it can be proved that if $x_0 \in \mathcal{C}$, then $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$. ■

Theorem 3.2 proves that if the reference trajectory verifies the constraints on the plant state, that is, $x_{\mathrm{ref}}(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, then the model reference adaptive control law (21) with adaptive laws (32)–(34) guarantees that the plant trajectory lays in the constraint set at all time, that is, $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, and, per definition of uniform asymptotic convergence (Haddad & Chellaboina, 2008, Definition 4.2), the plant trajectory $x(\cdot)$ approaches an arbitrarily small neighbourhood of the reference trajectory $x_{\mathrm{ref}}(\cdot)$ in finite-time, irrespectively of the initial time $t_0 \geq 0$. The positive-definite, decrescent function (25) is a *generalised barrier Lyapunov function* (Liu et al., 2017), since $V(\cdot,\cdot,\cdot,\cdot,\cdot)$ is employed to prove uniform asymptotic stability of the nonlinear dynamical system given by (23), and if $\lim_{t\to\overline{T}} \mathrm{dist}(x(t), \partial\mathcal{C}) = 0$ uniformly in t_0 for some $\overline{T} >$

t_0, then $\lim_{t\to\overline{T}} V(t, e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t)) = \infty$ uniformly in t_0. In the absence of a constraint set, Theorem 3.2 reduces to the classical model reference adaptive control framework (Lavretsky & Wise, 2012, pp. 281–286). Indeed, if $h(x) = 1$, $x \in \mathcal{D}$, then $\mathcal{C} = \mathcal{D}$ and (32)–(34) reduce to the classical adaptive laws for model reference adaptive control (Lavretsky & Wise, 2012, Theorem 9.2).

Remark 3.1: Theorem 3.2 can be applied to enforce at all time boundedness of the trajectory tracking error within margins assigned a priori.

To appreciate the significance of Remark 3.1, consider, for instance, the constraint set

$$\mathcal{C} = \left\{x \in \mathcal{D} : \eta - \|x - s_{\mathrm{ref}}(t, x_0)\|_R^2 \geq 0, \text{ for some } t \geq t_0\right\}, \tag{35}$$

where $\eta > 0$, $\|x\|_R = [x^{\mathrm{T}} R x]^{1/2}$, $x \in \mathbb{R}^n$, $R \in \mathbb{R}^{n\times n}$ is symmetric and positive-definite, and $s_{\mathrm{ref}}(t, x_{\mathrm{ref}})$, $(t, x_{\mathrm{ref}}) \in [t_0, \infty) \times \mathcal{D}$, denotes the solution of (10) with initial condition x_{ref} at time $t \geq t_0$ (Haddad & Chellaboina, 2008, p. 71); note that the trajectory of the reference model (10) is continuous and bounded, and hence $\mathcal{C}$ is a compact, simply connected subset of $\mathcal{D}$. Since (35) is in the same form as (12), Theorem 3.2 can be applied to design an adaptive control law so that $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, and hence to impose that $\eta > [x(t) - x_{\mathrm{ref}}(t)]^{\mathrm{T}}(t) R[x(t) - x_{\mathrm{ref}}(t)]$, $t \geq t_0$.

It is worthwhile to recall that classical model reference adaptive control guarantees boundedness of the trajectory tracking error (11) at all time. However, the bounds on the the trajectory tracking error, which are functions of the plant's unknown parameters, cannot be estimated (Lavretsky & Wise, 2012, pp. 281–285). Moreover, if the plant dynamics were perfectly known, then the bounds on the trajectory tracking error would be defined by the domain of attraction of the trajectory tracking error dynamics (23) (Haddad & Chellaboina, 2008, p. 142). However, there not exist analytical methods to characterise the domain of attraction of any nonlinear asymptotically convergent dynamical system; currently, there only exist analytical and numerical methods that provide conservative estimates of domains of attraction (Chesi, 2011; Ermolin & Vlasova, 2015; Hu, Goebel, Teel, & Lin, 2005; Matallana, Blanco, & Bandoni, 2010; Parrilo, 2000, Chapter 7). Finally, conventional model reference adaptive control only allows to *indirectly* regulate the bounds on the trajectory tracking error by choosing the symmetric positive-definite matrices Γ_x, Γ_r, Γ_Θ, and Q, and hence the symmetric positive-definite matrix P that verifies the Lyapunov equation (13). Theorem 3.2, instead, allows to directly impose *any* constraint on the trajectory tracking error that can be modelled as a compact, simply connected set in the same form as (12).

It follows from the proofs of Theorems 3.1 and 3.2 that the adaptive gains $\Delta K_x(\cdot)$, $\Delta K_r(\cdot)$, and $\Delta\Theta(\cdot)$ are bounded at all time. However, it follows from (29) and (30) that the bounds on the adaptation gains are functions of Λ, which is unknown. Thus, applying the framework presented in this section, the ultimate bounds on the adaptation gains can be neither imposed a priori. In the next section, we present alternatives to Theorems 3.1 and 3.2 that overcome this limitation.

4. Projection operator and restricted potential functions

In this section, we show an approach to solve the constrained output tracking problem, which employs the projection operator (Kreisselmeier & Narendra, 1982; Pomet & Praly, 1992) to enforce prescribed bounds on the adaptive gains. The control algorithms presented hereafter guarantee that the plant's measured output lays in a compact, simply connected constraint set and tracks a bounded reference signal with uniformly ultimately bounded error, even in case the output reference signal does not lay in the constraint set. Despite the approach presented in Section 3, the results presented hereafter allow us to impose bounds on the adaptation gains, which are assigned a priori. Moreover, the control laws presented in the following guarantee robustness not only to matched and parametric uncertainties, but also unmatched uncertainties.

In practical applications involving computers with limited system's memory, such as autopilots for small multi-rotor aircraft, it is useful to bound the adaptation gains within given margins. However, employing the projection operator introduces additional complexity in the control algorithm and further strains the performance of the devices computing the adaptation laws. Therefore, the choice among the framework presented in Section 3, the one presented in this section, or alternative frameworks deduced by applying, for instance, the dead-zone modification (Peterson & Narendra, 1982), the σ-modification (Ioannou & Kokotovic, 1983), and the μ-modification (Lavretsky & Hovakimyan, 2004) of classical model reference adaptive control (Lavretsky & Wise, 2012, pp. 281–291), also depends on the capabilities of the computers implementing the control algorithms.

Consider the nonlinear time-varying plant and the plant sensors' dynamics

$$\begin{aligned}\dot{x}_{\mathrm{p}}(t) &= A_{\mathrm{p}} x_{\mathrm{p}}(t) + B_{\mathrm{p}}\Lambda\left[u(t) + \Theta^{\mathrm{T}}\Phi(x_{\mathrm{p}}(t))\right] + \hat{\xi}(t),\\ x_{\mathrm{p}}(t_0) &= x_{\mathrm{p},0}, \quad t \geq t_0,\end{aligned} \tag{36}$$

$$\dot{y}(t) = \theta C_{\mathrm{p}} x_{\mathrm{p}}(t) - \theta y(t), \quad y(t_0) = C_{\mathrm{p}} x_{\mathrm{p},0}, \tag{37}$$

where $x_{\mathrm{p}}(t) \in \mathcal{D}_{\mathrm{p}} \subseteq \mathbb{R}^{n_{\mathrm{p}}}$, $t \geq t_0$, denotes the *plant's trajectory*, $0 \in \mathcal{D}_{\mathrm{p}}$, $u(t) \in \mathbb{R}^m$ denotes the *control input*, $y(t) \in \mathbb{R}^m$ denotes the *measured output*, $\theta > 0$, $A_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}\times n_{\mathrm{p}}}$ is *unknown*, $B_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}\times m}$, $C_{\mathrm{p}} \in \mathbb{R}^{m\times n_{\mathrm{p}}}$, $\Lambda \in \mathbb{R}^{m\times m}$ is diagonal, positive-definite, and *unknown*, $\Theta \in \mathbb{R}^{N\times m}$ is *unknown*, the *regressor vector* $\Phi : \mathbb{R}^{n_{\mathrm{p}}} \to \mathbb{R}^N$ is Lipschitz continuous in its argument, and $\hat{\xi} : [t_0, \infty) \to \mathbb{R}^{n_{\mathrm{p}}}$ is continuous in its argument and *unknown*. We assume that $\|\hat{\xi}(t)\| \leq \hat{\xi}_{\max}$, $t \geq t_0$, where $\hat{\xi}_{\max} > 0$ is unknown, and Λ is such that the pair $(A_{\mathrm{p}}, B_{\mathrm{p}}\Lambda)$ is controllable and $\Lambda_{\min} I_m \leq \Lambda$, for some $\Lambda_{\min} > 0$. Both Λ and $\Theta^{\mathrm{T}}\Phi(x_{\mathrm{p}})$, $x_{\mathrm{p}} \in \mathcal{D}_{\mathrm{p}}$, capture the plant's matched and parametric uncertainties, such as malfunctions in the control system; the term $\hat{\xi}(\cdot)$ captures the plant's unmatched uncertainties, such as external disturbances. Equation (37) models the plant sensors as linear dynamical systems, whose transient dynamics is characterised by the parameter θ (Morris, 2001, Chapter 2). Given the *output reference signal* $y_{\mathrm{cmd}} : [t_0, \infty) \to \mathbb{R}^m$, which is continuous with its first derivative, define $y_{\mathrm{cmd},2}(t) \triangleq \dot{y}_{\mathrm{cmd}}(t)$, $t \geq t_0$, and assume that both $y_{\rm cmd}(\cdot)$ and $y_{\rm cmd,2}(\cdot)$ are bounded, that is, $\|y_{\rm cmd}(t)\| \leq y_{\max,1}$, $t \geq t_0$, and $\|y_{\rm cmd,2}(t)\| \leq y_{\max,2}$, where $y_{\max,1}, y_{\max,2} \geq 0$. Consider also the compact, simply connected constraint set

$$\mathcal{C}_y \triangleq \left\{y \in \mathbb{R}^m : h_y(y) \geq 0\right\}, \tag{38}$$

where $h_y : \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable and $h_y(0) > 0$.

The next theorem is the main result of this section and provides a robust model reference adaptive control law for $u(\cdot)$ so that $y(t) \in \mathring{\mathcal{C}}_y$, $t \geq t_0$, and the *output tracking error* $e_y(t) \triangleq y(t) - y_{\rm cmd}(t)$, $t \geq t_0$, is uniformly ultimately bounded, that is, there exists $\varepsilon > 0$ and $\gamma > 0$ such that, for every $\delta \in (0, \gamma)$, there exists a finite-time $T = T(\delta, \varepsilon) \geq 0$ such that $e_y(t_0) \in \mathcal{B}_\delta(0)$ implies

$$e_y(t) \in \mathcal{B}_\varepsilon(0), \quad t \geq t_0 + T. \tag{39}$$

This control law is effective even in case $y_{\rm cmd}(t) \notin \mathcal{C}_y$ for some $t \in [t_0, \infty)$. For the statement of this result, let $\rho_{\mathcal{C}_y} \triangleq \max_{y \in \mathcal{C}_y} \|y\|$ and $\overline{h}_{y,\rm d} \triangleq \max_{y \in \mathcal{C}_y} \|h'_y(y)\|$. Furthermore, let $n \triangleq n_{\rm p} + m$ and $x(t) \triangleq [x_{\rm p}^{\rm T}(t), [y(t) - y_{\rm cmd}(t)]^{\rm T}]^{\rm T} \in \mathbb{R}^n$, $t \geq t_0$, note that (36) and (37) are equivalent to

$$\dot{x}(t) = Ax(t) + B\Lambda\left[u(t) + \Theta^{\rm T}\Phi(x_{\rm p}(t))\right] + \xi(t),$$
$$x(t_0) = \begin{bmatrix} x_{\rm p,0} \\ C_{\rm p}x_{\rm p,0} - y_{\rm cmd}(t_0)\end{bmatrix}, \quad t \geq t_0, \tag{40}$$

where

$$x(t) \in \mathcal{D} \subseteq \mathbb{R}^n, \quad \mathcal{D} \triangleq \mathcal{D}_{\rm p} \times \mathbb{R}^m, \quad A \triangleq \begin{bmatrix} A_{\rm p} & 0_{n_{\rm p}\times m} \\ \theta C_{\rm p} & -\theta I_m\end{bmatrix},$$
$$B \triangleq \begin{bmatrix} B_{\rm p} \\ 0_{m\times m}\end{bmatrix}, \quad B_1 \triangleq \begin{bmatrix} 0_{n_{\rm p}\times m} \\ -I_m\end{bmatrix},$$

and $\xi(t) \triangleq B_1[y_{\rm cmd,2}(t) + \theta y_{\rm cmd}(t)] + \begin{bmatrix} I_{n_{\rm p}} \\ 0_{m\times n_{\rm p}}\end{bmatrix}\hat{\xi}(t)$, and consider the *reference dynamical model*

$$\dot{x}_{\rm ref}(t) = A_{\rm ref}x_{\rm ref}(t) + B_{\rm ref}y_{\rm cmd}(t), \quad x_{\rm ref}(t_0) = x_{\rm ref,0}, \quad t \geq t_0, \tag{41}$$

where $A_{\rm ref} = \begin{bmatrix} A_{\rm ref,1} & 0_{n_{\rm p}\times m} \\ 0_{m\times n_{\rm p}} & A_{\rm ref,2}\end{bmatrix}$, both $A_{\rm ref,1} \in \mathbb{R}^{n_{\rm p}\times n_{\rm p}}$ and $A_{\rm ref,2} \in \mathbb{R}^{m\times m}$ are Hurwitz, and $B_{\rm ref} \in \mathbb{R}^{n\times m}$. Lastly, let $h(x(t)) = h_y(-B_1x(t) + y_{\rm cmd}(t))$, $t \geq t_0$, where $x(\cdot)$ denotes the solution of (40), and $\mathcal{C} = \{x \in \mathcal{D} : h(x) \geq 0\}$.

Theorem 4.1: *Consider the nonlinear plant* (36) *with measured output dynamics* (37)*, the constraint set* (38)*, the augmented dynamical system* (40)*, the reference model* (41)*, and the adaptive laws*

$$\dot{\hat{K}}_x(t) = \mathrm{Proj}\left(\hat{K}_x(t), -\Gamma_x x(t)\frac{e^{\rm T}(t)P}{h(x(t))}\left[I_n - \frac{e(t)h'(x(t))}{2\,h(x(t))}\right]B\right),$$
$$\hat{K}_x(t_0) = \hat{K}_{x,0}, \quad t \geq t_0, \tag{42}$$

$$\dot{\hat{K}}_r(t) = \mathrm{Proj}\left(\hat{K}_r(t), -\Gamma_r y_{\rm cmd}(t)\frac{e^{\rm T}(t)P}{h(x(t))}\left[I_n - \frac{e(t)h'(x(t))}{2\,h(x(t))}\right]B\right),$$
$$\hat{K}_r(t_0) = \hat{K}_{r,0}, \tag{43}$$

$$\dot{\hat{\Theta}}(t) = \mathrm{Proj}\left(\hat{\Theta}(t), \Gamma_\Theta\Phi(x(t))\frac{e^{\rm T}(t)P}{h(x(t))}\left[I_n - \frac{e(t)h'(x(t))}{2\,h(x(t))}\right]B\right),$$
$$\hat{\Theta}(t_0) = \hat{\Theta}_0, \tag{44}$$

where κ is given by (14)*, the matrices $\Gamma_x \in \mathbb{R}^{n\times n}$, $\Gamma_r \in \mathbb{R}^{m\times m}$, and $\Gamma_\Theta \in \mathbb{R}^{N\times N}$ are symmetric and positive-definite, $e(t) = x(t) - x_{\rm ref}(t)$, $t \geq t_0$, P is the symmetric positive-definite solution of the Lyapunov equation* (13)*, and* $\mathrm{Proj}(\cdot,\cdot)$ *is given by* (6)*. If $y(t_0) \in \mathring{\mathcal{C}}_y$,*

$$\frac{\lambda_{\max}(P)\overline{h}_{y,\rm d}\left[\|A_{\rm ref}\|\rho_{\mathcal{C}_y} + \|B_{\rm ref}\|y_{\max,1}\right]}{\lambda_{\min}(Q)} < \kappa, \tag{45}$$

where κ is given by (14)*, and there exist $K_x \in \mathbb{R}^{n\times m}$ and $K_r \in \mathbb{R}^{m\times m}$ such that* (19) *and* (20) *are verified, then the trajectory of* (36) *and* (37) *with $u = \phi(t, x, \hat{K}_x, \hat{K}_{\rm cmd}, \hat{\Theta})$, where*

$$\phi(t, x, \hat{K}_x, \hat{K}_r, \hat{\Theta}) = \hat{K}_x^{\rm T}x + \hat{K}_r^{\rm T}y_{\rm cmd}(t) - \hat{\Theta}^{\rm T}\Phi(x),$$
$$(t, x, \hat{K}_x, \hat{K}_r, \hat{\Theta}) \in [t_0, \infty) \times \mathring{\mathcal{C}} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times m}, \tag{46}$$

is such that $y(t) \in \mathring{\mathcal{C}}_y$, $t \geq t_0$. Furthermore, there exist $\varepsilon > 0$ and $\gamma > 0$ such that, for every $\delta \in (0, \gamma)$, there exists a finite-time $T = T(\delta, \varepsilon) \geq 0$ such that $e_y(t_0) \in \mathcal{B}_\delta(0)$ implies (39).

Proof: The proof of this result is similar to the proof of Theorem 3.1 and hence, is only briefly outlined. Let $\Delta K_x = \hat{K}_x - K_x$, $\Delta K_r = \hat{K}_r - K_r$, and $\Delta\Theta = \hat{\Theta} - \Theta$. Assuming that $y(t) \in \mathring{\mathcal{C}}_y$, $t \geq t_0$, uniform ultimate boundedness of the trajectory tracking error can be proved considering the positive-definite and decrescent function

$$V(t, e, \Delta K_x, \Delta K_r, \Delta\Theta)$$
$$\triangleq \frac{e^{\rm T}Pe}{h_y(y(t))} + \mathrm{tr}\left(\left[\Delta K_x^{\rm T}\Gamma_x^{-1}\Delta K_x + \Delta K_r^{\rm T}\Gamma_r^{-1}\Delta K_r + \Delta\Theta^{\rm T}\Gamma_\Theta^{-1}\Delta\Theta\right]\Lambda\right),$$
$$(t, e, \Delta K_x, \Delta K_r, \Delta\Theta) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times m}, \tag{47}$$

and applying Theorem 2.2 to show that

$$\dot{V}(t, e, \Delta K_x, \Delta K_r, \Delta\Theta) \leq -W(e, \Delta K_x, \Delta K_r, \Delta\Theta),$$
$$(t, e, \Delta K_x, \Delta K_r, \Delta\Theta) \in [t_0, \infty) \times \mathcal{Q}, \tag{48}$$

along the trajectories of (42)–(44) and

$$\dot{e}(t) = A_{\rm ref}e(t) + B\Lambda\left[\Delta K_x^{\rm T}(t)x(t) + \Delta K_r^{\rm T}(t)r(t)\right.$$

$$- \Delta\Theta^{\mathrm{T}}(t)\Phi(x(t))\Big] + B_1\xi(t),$$

$$e(t_0) = x_0 - x_{\mathrm{ref},0}, \quad t \geq t_0, \tag{49}$$

where

$$\begin{aligned} &W(e, \Delta K_x, \Delta K_r, \Delta\Theta) \\ &= -\kappa^{-2}\Big[\lambda_{\max}(P)\overline{h}_{y,\mathrm{d}}\left(\|A_{\mathrm{ref}}\|\rho_{\mathcal{C}_y} + \|B_{\mathrm{ref}}\|y_{\max,1}\right) \\ &\quad -\lambda_{\min}(Q)\kappa\Big]\|e\|^2 \\ &\quad - 2\kappa^{-1}\lambda_{\max}(P)\hat{\xi}_{\max}\|e\|, \\ &\mathcal{Q} = \Big\{(e, \Delta K_x, \Delta K_r, \Delta\Theta) \in \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \\ &\quad \times \mathbb{R}^{N\times m} : \|e\| > c \\ &\quad \text{and } \|\Delta K_x\|_{\mathrm{F}} > 2n\overline{\theta} \\ &\quad \text{and } \|\Delta K_r\|_{\mathrm{F}} > 2\,m\overline{\theta} \text{ and } \|\Delta\Theta\|_{\mathrm{F}} > 2N\overline{\theta}\Big\}, \end{aligned}$$

and $\overline{\theta} > 0$ verifies (4); an explicit expression for $c > 0$ is omitted for brevity. Since (47) is positive-definite and decrescent, there exist class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that (31) is verified. Thus, it follows from (47) and (48) that the conditions of Theorem 2.1 are verified and if $y(t) \in \mathring{\mathcal{C}}_y$, then the nonlinear time-invariant dynamical system given by (49) and (42)–(44) is uniformly ultimately bounded.

Next, let $x_{\mathrm{ref}}(t) = [x_{\mathrm{ref},1}^{\mathrm{T}}(t), x_{\mathrm{ref},2}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $t \geq t_0$, verify (41), where $x_{\mathrm{ref},1}(t) \in \mathbb{R}^{n_{\mathrm{p}}}$ and $x_{\mathrm{ref},2}(t) \in \mathbb{R}^m$. It follows from the uniform ultimate boundedness of (49) and (42)–(44) that there exist $\hat{b} > 0$ and $\gamma > 0$ such that for every $\delta \in (0, \gamma)$, there exists a finite-time $T = T(\delta, \varepsilon) \geq 0$ such that if $e_y(t_0) \in \mathcal{B}_\delta(0)$, then

$$\left\|y(t) - y_{\mathrm{cmd}}(t) - x_{\mathrm{ref},2}(t)\right\| \leq \hat{b}, \quad t \geq T + t_0. \tag{50}$$

Moreover, since A_{ref} is block-diagonal and Hurwitz, $B_1 = [0_{m\times n_{\mathrm{p}}}, -I_m]^{\mathrm{T}}$, and $y_{\mathrm{cmd}}(\cdot)$ is bounded, it follows from (41) that $x_{\mathrm{ref},2}(\cdot)$ is uniformly bounded (Haddad & Chellaboina, 2008, p. 245), that is, $\|x_{\mathrm{ref},2}(t)\| \leq b_2$, $t \geq t_0$, for some $b_2 \geq 0$ independent of t_0. Thus, it follows from (50) that there exists $\gamma > 0$ such that for every $\delta \in (0, \gamma)$, there exists a finite-time $T = T(\delta, \varepsilon) \geq 0$ such that if $e_y(t_0) \in \mathcal{B}_\delta(0)$, then (39) is verified with $\varepsilon = \hat{b} + b_2$. Lastly, using a contradiction argument similar to the one presented in the proof of Theorem 3.1, one can prove that if $y_0 \in \mathcal{C}_y$, then $y(t) \in \mathring{\mathcal{C}}_y$, $t \geq t_0$. ■

The next corollary provides sufficient conditions for the output tracking error to verify prescribed bounds. For the statement of this result, consider the compact, simply connected constraint set

$$\mathcal{C}_y = \left\{y \in \mathbb{R}^m : \eta - \left[y - y_{\mathrm{cmd}}(\cdot)\right]^{\mathrm{T}} R\left[y - y_{\mathrm{cmd}}(\cdot)\right] \geq 0\right\}, \tag{51}$$

where $\eta > 0$ and R is symmetric and positive-definite.

Corollary 4.1: *Consider the nonlinear plant given by (36) and (37) with $y(t_0) \in \mathring{\mathcal{C}}_y$, the constraint set (51), the augmented dynamical system (40), the reference model (41), and the adaptive laws (42)–(44). If there exist $K_x \in \mathbb{R}^{n\times m}$ and $K_r \in \mathbb{R}^{m\times m}$ such that (19) and (20) are verified, then the measured output of (36) and (37) with $u = \phi(t, x, \hat{K}_x, \hat{K}_r, \hat{\Theta})$, where $\phi(\cdot,\cdot,\cdot,\cdot,\cdot)$ is given by (46), is such that $y(t) \in \mathring{\mathcal{C}}_y$, $t \geq t_0$.*

Proof: The result follows as in the proof of Theorem 3.2 and is omitted for brevity. ■

If the measured output reference signal lays in the constraint set (51) at all time, then Corollary 4.1 allows to impose bounds on the output tracking error, which are captured by η and R. This result is similar to Theorem 3.1 of (Arabi et al., 2018).

5. Illustrative numerical example

In this section, we illustrate the applicability of our theoretical framework by solving the constrained attitude tracking control problem for a rigid spacecraft, whose principal moments of inertia are the same as the Cassini spacecraft's (Lorenz, 2017). For its ability to enforce constraints on the vehicle's rotational dynamics, the control law presented hereafter is applicable to those space vehicles, whose payload must avoid pointing towards some specific directions, such as the sun.

Let $\psi : [0, \infty) \to \mathbb{R}$, $\theta : [0, \infty) \to (-(\pi/2), (\pi/2))$, and $\phi : [0, \infty) \to \mathbb{R}$ denote the spacecraft's yaw, pitch, and roll angles, respectively, and $p, q, r : [0, \infty) \to \mathbb{R}$ denote the components of the vehicle's angular velocity expressed in the principal reference frame. In this case, the spacecraft's rotational equations of motion are given by Greenwood (2003, Chapter 3)

$$\begin{bmatrix} \dot{\phi}(t) \\ \dot{\theta}(t) \\ \dot{\psi}(t) \\ \dot{p}(t) \\ \dot{q}(t) \\ \dot{r}(t) \end{bmatrix} = f(\phi(t), \theta(t), \psi(t), p(t), q(t), r(t)) + \begin{bmatrix} 0 \\ \hat{J}^{-1} \end{bmatrix} v(t) + \hat{\xi}(t), \quad \begin{bmatrix} \phi(0) \\ \theta(0) \\ \psi(0) \\ p(0) \\ q(0) \\ r(0) \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \theta_0 \\ \psi_0 \\ p_0 \\ q_0 \\ r_0 \end{bmatrix}, \quad t \geq 0, \tag{52}$$

where

$$f(\phi(t), \theta(t), \psi(t), p(t), q(t), r(t)) = \begin{bmatrix} p(t) + q(t)\sin\phi(t)\tan\theta(t) + r(t)\cos\phi(t)\tan\theta(t) \\ q(t)\cos\phi(t) - r(t)\sin\phi(t) \\ q(t)\sin\phi(t)\sec\theta(t) + r(t)\cos\phi(t)\sec\theta(t) \\ -\hat{J}^{-1}\begin{bmatrix} 0 & -r(t) & q(t) \\ r(t) & 0 & -p(t) \\ -q(t) & p(t) & 0 \end{bmatrix}\hat{J}\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} \end{bmatrix}, \tag{53}$$

$\hat{J} \in \mathbb{R}^{3\times3}$ denotes the spacecraft's estimated matrix of inertia, $v : [0,\infty) \to \mathbb{R}^3$ denotes the control vector, $\hat{\xi}(t) = [0^{\mathrm{T}}, \hat{\xi}_2^{\mathrm{T}}(t)]^{\mathrm{T}}$,

$$\begin{aligned}\hat{\xi}_2(t) = \hat{J}^{-1}\begin{bmatrix}0 & -r(t) & q(t)\\ r(t) & 0 & -p(t)\\ -q(t) & p(t) & 0\end{bmatrix}\hat{J}\begin{bmatrix}p(t)\\ q(t)\\ r(t)\end{bmatrix}\\ - J^{-1}\begin{bmatrix}0 & -r(t) & q(t)\\ r(t) & 0 & -p(t)\\ -q(t) & p(t) & 0\end{bmatrix}J\begin{bmatrix}p(t)\\ q(t)\\ r(t)\end{bmatrix}\\ + \left(J - \hat{J}\right) v(t),\end{aligned} \tag{54}$$

and $J \in \mathbb{R}^{3\times3}$ denotes the spacecraft's matrix of inertia.

By proceeding as in Example 6.3 of L'Afflitto and Haddad (2016), one can prove that the nonlinear dynamical system (52) is feedback linearisable (Isidori, 1995, Chapter 5). Specifically, (52) with

$$\begin{aligned}v(t) = \hat{J}\begin{bmatrix}1 & 0 & -\sin\theta(t)\\ 0 & \cos\phi(t) & \cos\theta(t)\sin\phi(t)\\ 0 & -\sin\phi(t) & \cos\theta(t)\cos\phi(t)\end{bmatrix}\\ \times\left(\begin{bmatrix}\dot{\phi}(t)\\ \dot{\theta}(t)\\ \dot{\psi}(t)\end{bmatrix} - \beta(\phi(t),\theta(t),\psi(t)) + u(t)\right), \quad t \geq 0,\end{aligned} \tag{55}$$

is equivalent to

$$\dot{x}_{\mathrm{p}}(t) = A_{\mathrm{p}}x_{\mathrm{p}}(t) + B_{\mathrm{p}}\Lambda\left[u(t) + \Theta^{\mathrm{T}}\Phi(x_{\mathrm{p}}(t))\right] + \hat{\xi}(t),$$
$$x_{\mathrm{p}}(0) = \left[\phi_0, \dot{\phi}_0, \theta_0, \dot{\theta}_0, \psi_0, \dot{\psi}_0\right]^{\mathrm{T}}, \quad t \geq 0, \tag{56}$$

$$\dot{y}(t) = \varepsilon C_{\mathrm{p}}x_{\mathrm{p}}(t) - \varepsilon y(t), \quad y(0) = C_{\mathrm{p}}x_{\mathrm{p}}(0), \tag{57}$$

where $\beta(\phi,\theta,\psi) \triangleq [L_f^2\phi, L_f^2\theta, L_f^2\psi]^{\mathrm{T}}$, $L_f^2(\cdot)$ denotes the second Lie derivative of its argument along the vector field $f(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ (Haddad & Chellaboina, 2008, Definition 6.8), $u : [0,\infty) \to \mathbb{R}^3$, $x_{\mathrm{p}}(t) = [\phi(t), \dot{\phi}(t), \theta(t), \dot{\theta}(t), \psi(t), \dot{\psi}(t)]^{\mathrm{T}}$,

$$A_{\mathrm{p}} = \begin{bmatrix}0&1&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&-1&0&0\\0&0&0&0&0&1\\0&0&0&0&0&-1\end{bmatrix}, \quad B_{\mathrm{p}} = \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1\end{bmatrix},$$

$$C_{\mathrm{p}} = \begin{bmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1\\0&0&0\end{bmatrix}^{\mathrm{T}}, \tag{58}$$

$$\begin{bmatrix}\dot{\phi}_0\\ \dot{\theta}_0\\ \dot{\psi}_0\end{bmatrix} = \begin{bmatrix}1 & 0 & -\sin\theta_0\\ 0 & \cos\phi_0 & \cos\theta_0\sin\phi_0\\ 0 & -\sin\phi_0 & \cos\theta_0\cos\phi_0\end{bmatrix}\begin{bmatrix}p_0\\ q_0\\ r_0\end{bmatrix}, \tag{59}$$

$\Lambda \in \mathbb{R}^{3\times3}$ is diagonal positive-definite, $\Phi(x_{\mathrm{p}}) = \tanh x_{\mathrm{p}}$, $x_{\mathrm{p}} \in \mathbb{R}^6$, $\Theta \in \mathbb{R}^{6\times3}$, and $\varepsilon > 0$. Although $\Lambda = I$ (L'Afflitto & Haddad, 2016), we assume that Λ is *unknown*, $\Lambda_{\min}I_m \leq \Lambda$, for some $\Lambda_{\min} > 0$, and the pair $(A_{\mathrm{p}}, B_{\mathrm{p}}\Lambda)$ is controllable; the matrix Λ accounts for malfunctions of the propulsion system and erroneous modelling assumptions. Equation (57) captures the dynamics of the sensors measuring $\phi(\cdot)$, $\theta(\cdot)$, and $\psi(\cdot)$ (Morris, 2001, Chapter 2).

Our goal is to design a robust model reference adaptive control law for the virtual control input $u(\cdot)$ such that the measured output $y(\cdot)$ tracks the reference signal $y_{\mathrm{cmd}}(t) = [\phi_{\mathrm{ref}}(t), \theta_{\mathrm{ref}}(t), \psi_{\mathrm{ref}}(t)]^{\mathrm{T}}$, $t \geq 0$, with bounded error and $y(t) \in \mathcal{C}_y$, where

$$\mathcal{C}_y = \left\{y \in \mathbb{R}^3 : \eta - y^{\mathrm{T}}Ry \geq 0\right\}, \tag{60}$$

$\eta > 0$, and $R \in \mathbb{R}^{3\times3}$ is symmetric and non-negative-definite. Since the dynamical system given by (56) and (57) is in the same form as (36) and (37) with $n_{\mathrm{p}} = 6$, $m = 3$, $N = 6$, and $t_0 = 0$, and the constraint set (60) is in the same form as (38) with $h_y(y) = \eta - y^{\mathrm{T}}Ry$, $y \in \mathbb{R}^3$, this goal can be met by applying Theorem 4.1. Specifically, if (45) is verified and there exist $K_x \in \mathbb{R}^{9\times3}$ and $K_r \in \mathbb{R}^{3\times3}$ such that (19) and (20) are satisfied, then it follows from Theorem 4.1 that the adaptive control law (46) with adaptive laws (42)–(44) guarantees that $y(t) \in \mathring{\mathcal{C}}_y$, $t \geq 0$, and the output tracking error is uniformly ultimately bounded.

Let

$$J = \begin{bmatrix}4.7111 & 0 & 0\\ 0 & 8.1389 & 0\\ 0 & 0 & 8.8399\end{bmatrix}\cdot 10^3\ \mathrm{kg\,m^2}$$

denote the spacecraft's matrix of inertia, let

$$\hat{J} = \begin{bmatrix}4.7090 & 0 & 0\\ 0 & 8.1370 & 0\\ 0 & 0 & 8.8370\end{bmatrix}\cdot 10^3\ \mathrm{kg\,m^2}$$

denote the estimated inertia matrix,

$$R = \begin{bmatrix}0.5 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 0\end{bmatrix},$$

and $\eta = 0.75$; note that J captures the principal moments of inertia of the Cassini spacecraft (Lee & Wertz, 2002). Figure 1 shows a phase-space representation of the spacecraft's pitch and roll angles obtained applying (55) with $u = \phi(t, x, \hat{K}_x, \hat{K}_r, \hat{\Theta}) = \hat{K}_x^{\mathrm{T}}x + \hat{K}_r^{\mathrm{T}}y_{\mathrm{cmd}}(t) - \hat{\Theta}^{\mathrm{T}}\Phi(x)$, $(t, x, \hat{K}_x, \hat{K}_r, \hat{\Theta}) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{N\times m}$, and adaptive laws (42)–(44). Figure 1 also shows a phase-space representation of the reference pitch angle $\phi_{\mathrm{ref}}(\cdot)$ and the reference roll angle $\theta_{\mathrm{ref}}(\cdot)$, which form a circle of radius 0.52 spanned at an angular velocity of 0.15 Hz, and the boundary of the constraint set (60).

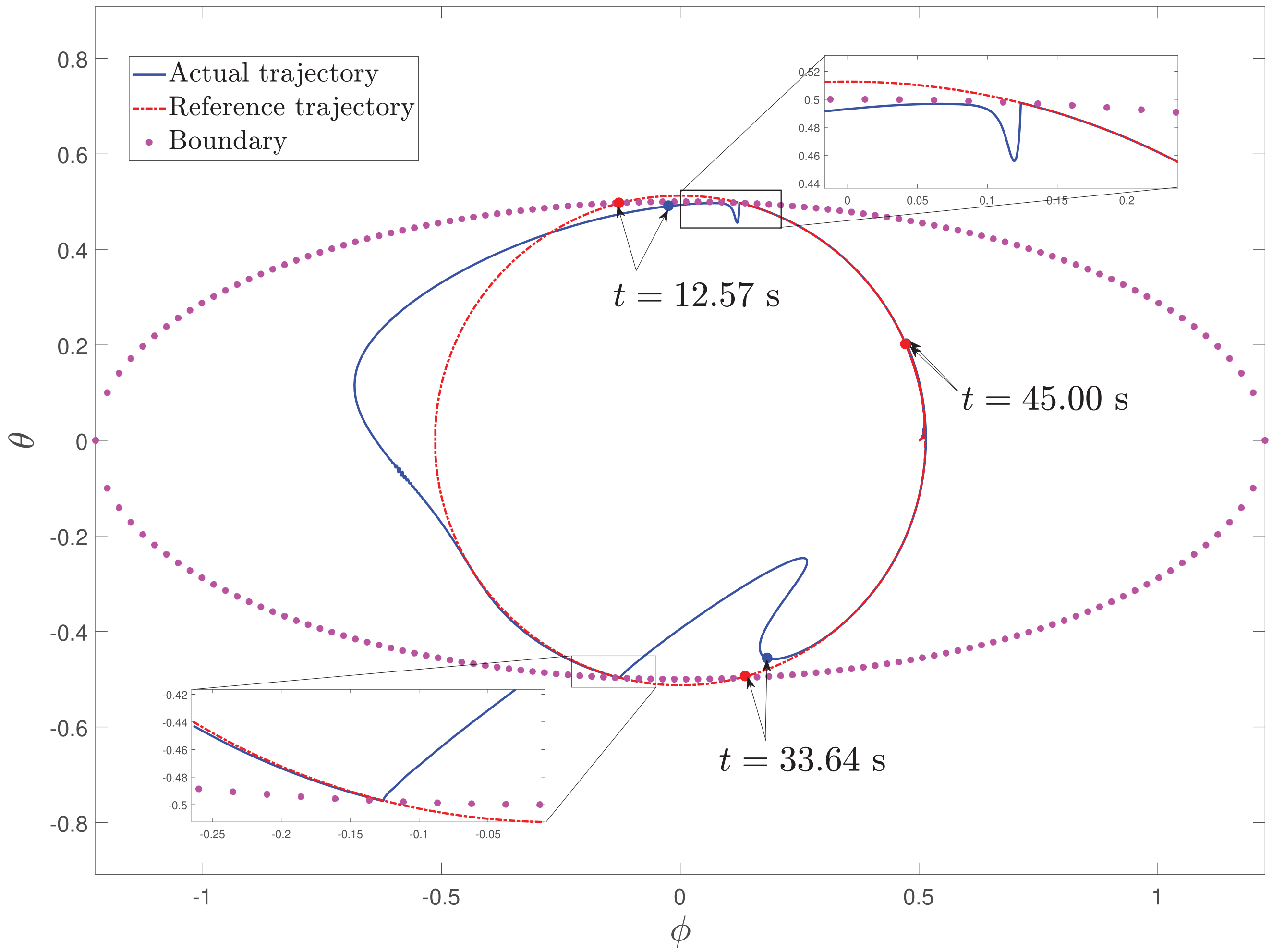

Figure 1. Phase-space representation of the spacecraft's pitch and roll angles.

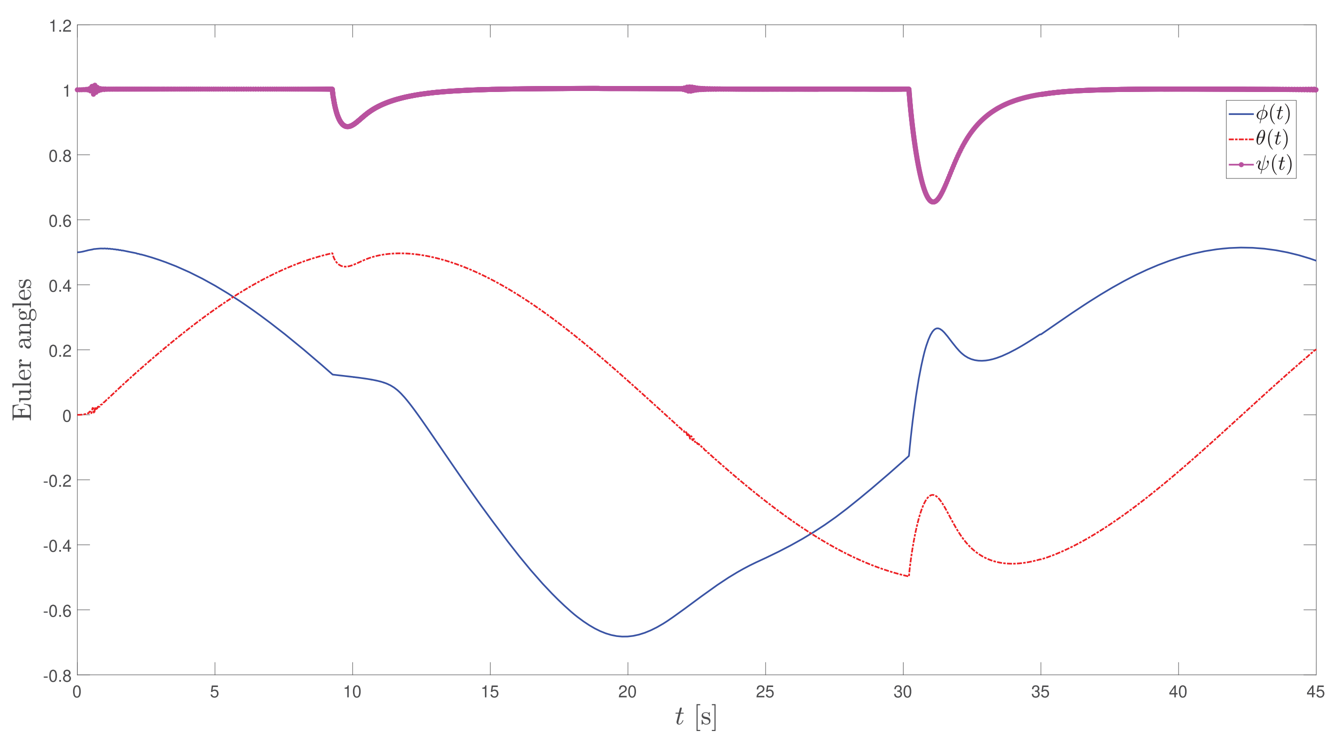

Figure 2. Spacecraft's yaw, pitch, and roll angles as functions of time.

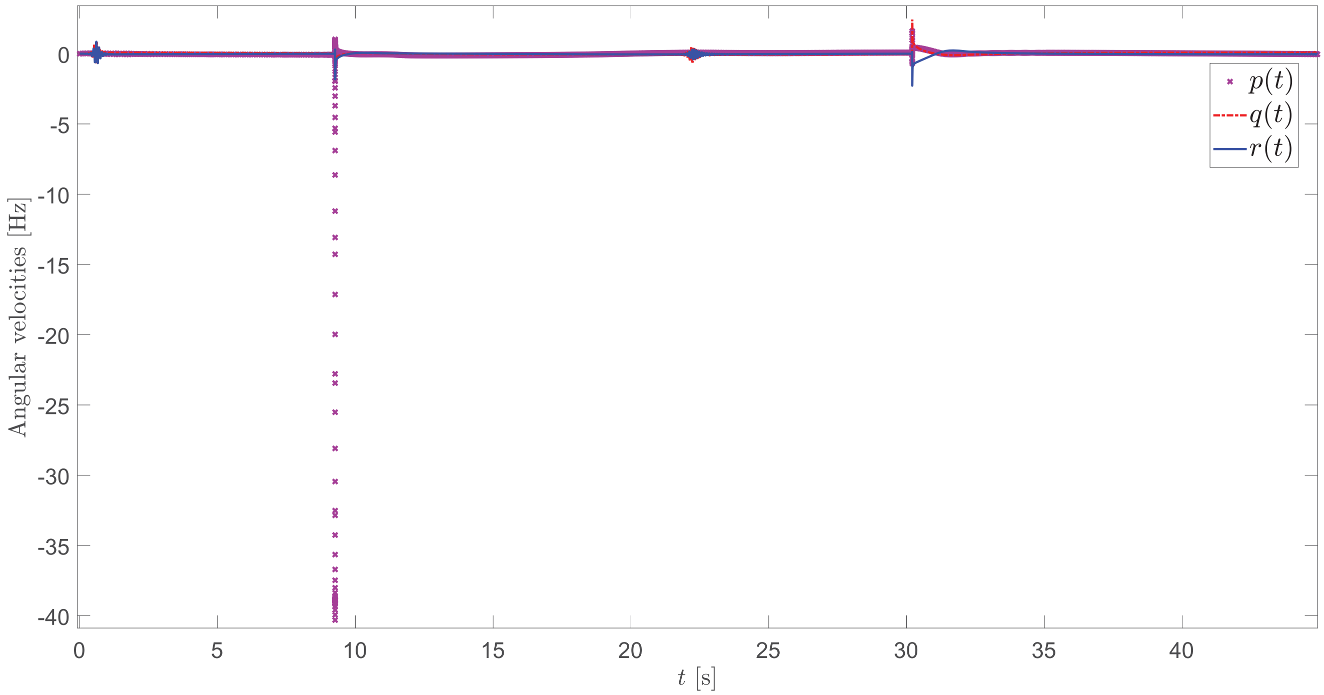

Figure 3. Components of the spacecraft's angular velocity as functions of time.

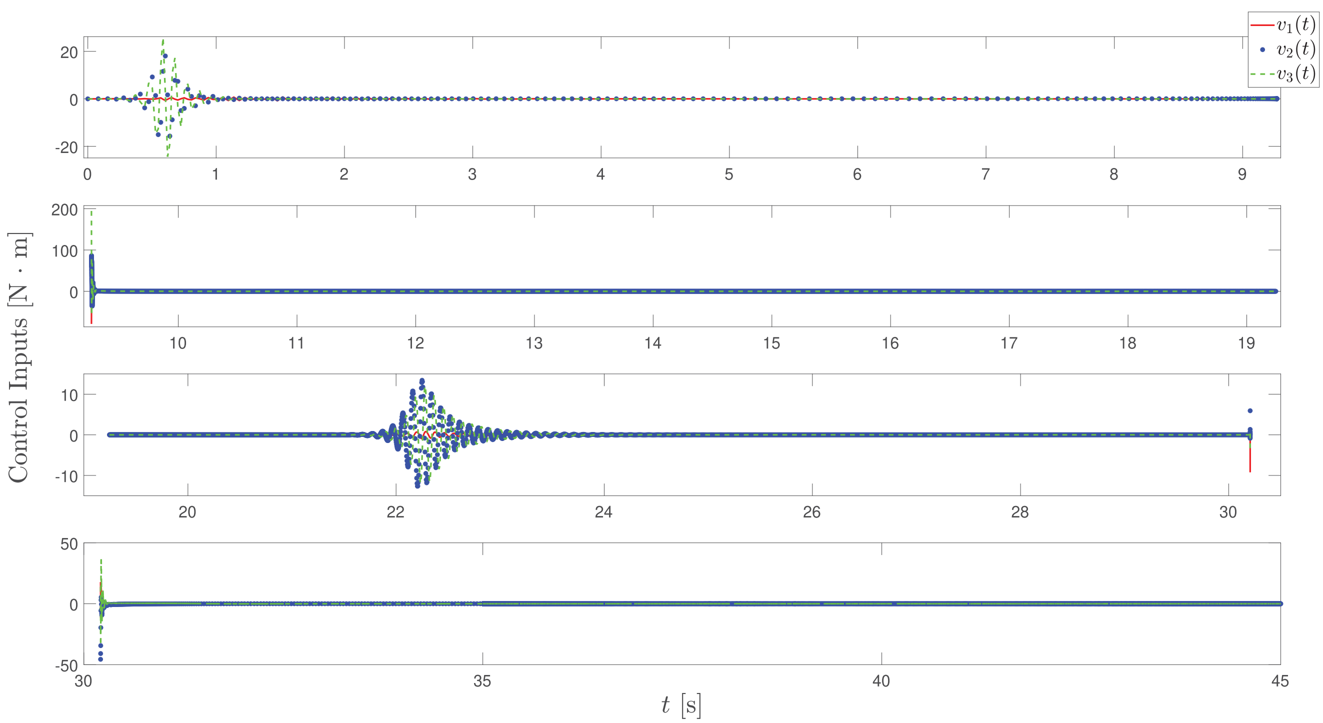

Figure 4. Control inputs as functions of time.

Figure 2 shows the spacecraft's yaw, pitch, and roll angles as functions of time, Figure 3 shows the components of the spacecraft's angular velocity as functions of time, and Figure 4 shows the components of the control input (55) as functions of time.

At $t = 9.26$ s, the spacecraft's pitch and roll angles approach the boundary of the constraint set for the first time and, for all $t \in [9.26, 12.57]$ s, the spacecraft's reference pitch and roll angles exceed the given constraints, whereas the vehicle's pitch and roll angles lay in the interior of the constraint set $\mathcal{C}_y$. At $t = 30.21$ s, the spacecraft's pitch and roll angles newly approach the boundary of the constraint set and once again, the control law (55) steers the spacecraft's rotational dynamics, so that the pitch and roll angles remain in the interior of $\mathcal{C}_y$. At $t = 33.64$ s, the reference pitch and roll angles newly verify the given constraints and the spacecraft pitch, and roll angles rapidly converge to their reference values. Both at $t = 9.26$ s and $t = 30.21$ s, the control inputs are practically impulsive. At all other times, the spacecraft control system is required to deliver a control moment comprised in the interval $[-25, 25]$ N · m. Both these impulsive control inputs and these continuous-time control moments can be delivered by conventional thrusters, such as those installed on the Cassini spacecraft (Burk, 2013; Wertz & Larson, 1999, Chapter 17).

6. Conclusion

In this paper, we provided model reference adaptive control laws for constrained nonlinear dynamical systems that guarantee robustness to uncertainties both in the plant model and the reference signal tracked by the plant trajectory or the measured output. Specifically, given a nonlinear plant, whose dynamics is affected by matched, unmatched, and parametric uncertainties, our control laws guarantee that the closed-loop system's trajectory and measured output track some reference signals with uniformly ultimately bounded error and are constrained to some compact, simply constraint set at all time. A unique feature of the proposed control laws is their effectiveness even in case the reference signals violate the given constraints and hence, may draw the plant trajectory and the measured output outside their constraint sets.

Future work directions concern a comparative analysis of the performance of plants implementing the proposed control laws with the performance of plants implementing a model predictive control law, a reference governor, and an explicit reference governor. Additional work directions involve the introduction of constraints on the control input and the implementation of these theoretical results on quadrotor helicopters involved in proximity operations.

Disclosure statement

No potential conflict of interest was reported by the authors.

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